2021 MANITOBA MATHEMATICAL CONTEST

For students in grade 12
9:00 AM – 11:00 AM
Tuesday, February 23, 2021

Manitoba Association of Mathematics Teachers

Winnipeg Actuaries' Club

Sponsored by:

The Winnipeg Actuaries' Club

The Manitoba Association of Mathematics Teachers

The Canadian Mathematical Society

The University of Manitoba

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Questions are found on both sides of this sheet. Answer as many as possible, but you are not expected to answer them all. **CALCULATORS ARE NOT PERMITTED**. Numerical answers by themselves, without explanation, will not receive full credit.

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1. (a) What is the probability that a randomly chosen number between 1 and 50 inclusive will have a 4 as a digit?

   (b) Let $X = |(-100) \cdot (-99) \cdots (99) \cdot (100)|$ (the absolute value of the product of all the integers from -100 to 100), and let $Y = (-100)^2 + (-99)^2 + \cdots + 99^2 + 100^2$. Which number is larger, X or Y? (mention a reason).

2. Find the area enclosed by the graph of

   (a) $x^2 + y^2 = 9$

   (b) $|x| + |y| = 3$

3. (a) If y is 2 more than x, and

$$z = \frac{\frac{y}{x} + \frac{x}{y} + 2}{\frac{y}{x} - \frac{x}{y}}$$

   then how much more than x is z?

   (b) Solve for x:

$$\frac{x^3 + x - 2}{x - 1} = 5$$

4. (a) Apollo runs in a race with 10 runners where 5 distinct trophies (1st place, 2nd place etc.) are given to the top 5 winners, in how many different ways can the prizes be given if there are no ties, and Apollo must be one of the top 3 winners?

   (b) If n is a positive integer and $2n + 1$ is a perfect square, show that $n + 1$ is the sum of two consecutive perfect squares.

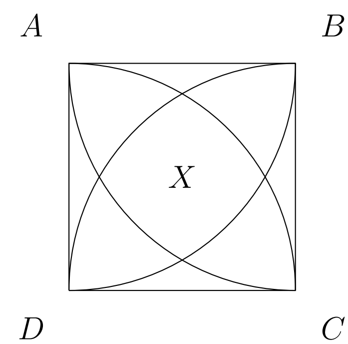
5. Solve the equation
$$\sqrt{x} + 3\sqrt[6]{x} = 4\sqrt[3]{x}.$$

6. Consider the set of numbers x satisfying $0 \leq x \leq 1$. Two numbers are randomly chosen from this interval. What is the probability that they differ by less than 0.2?

7. A wheel with radius 2 meters rolls around the outside of a regular convex heptagon* of side length 3 meters. Determine the length of the path traced out by the centre of the wheel.

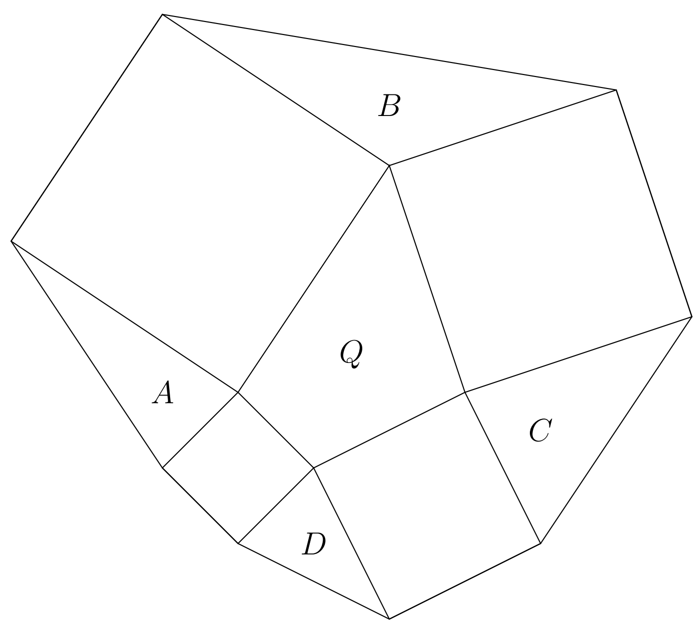
   ***regular heptagon**: a 7-sided polygon with equal sides and angles.

8. $\square ABCD$ is a square with sides of length 2. At each vertex a quarter circle of radius 2 is drawn as shown. Find the area of the intersection, X, of the four semicircles.



9. On each side of a convex* quadrilateral with area Q a square is constructed. Segments are added between near vertices of neighbouring squares, creating a figure consisting of Q, surrounded by four squares, alternating in cyclic order with four triangles of areas A, B, C and D (see diagram). Prove that $A + C = B + D = Q$.

   ***convex**: R is a convex region if any line segment whose endpoints are contained in R is entirely contained in R.



10. Find all ordered triples of polynomials $p(x), q(x), r(x)$ so that the following conditions hold:
    - $p(x)q(x)r(x) = (1 + x + x^2 + x^3 + x^4 + x^5)^3$
    - $p(1) = q(1) = r(1)$
    - $\deg(p) \leq \deg(q) \leq \deg(r)$ (deg(f) denotes the degree of polynomial $f(x)$)